Stochastic Iterative Learning Control Algorithms

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Stochastic ILC Laws

• Consider a discrete linear system in the ILC setting

\[ x_k(p + 1) = Ax_k(p) + B_k u(p) + \omega_k(p) \]  
\[ y_k(p) = Cx_k(p) + \nu_k(p), \quad 0 \leq t \leq T \]  

• where \( \omega_k(t) \) is the state disturbance and \( \nu_k(p) \) is the measurement error or noise

• The following ILC laws are considered

D-type:
\[ u_{k+1}(p) = u_k(p) + K_k [e_k(p + 1) - e_k(p)] \]  
P-type:
\[ u_{k+1}(p) = u_k(p) + K_k e_k(p + 1) \]  

where

\[ K_k = P_{u,k}(CB)^T \cdot [(CB)P_{u,k}(CB)^T + (C - CA)P_{x,k}(C - CA)^T + CQ_pC^T + R_p]^{-1} \]  
\[ P_{u,k+1} = (I - K_k CB)P_{u,k} \]  
\[ P_{x,k+1} = A P_{x,k} A^T + B P_{u,k} B^T + Q_p \]
Stochastic ILC laws

- $K_k$ is the ILC gain matrix

\[ Q_p = E[\omega(p, k)\omega^T(p, k)] \geq 0 \tag{8} \]

\[ P_{x,0} = E_k[\delta x(0, k)\delta x^T(0, k)] \geq 0 \tag{9} \]

(positive semi-definite matrices)

\[ R_t = E_k[\nu(0, k)\nu(0, k)^T] > 0 \tag{10} \]

(positive-definite matrix)

\[ P_{u,0} = E[\delta u(p, 0)\delta u(p, 0)^T] \tag{11} \]

is a positive-definite matrix
Stochastic ILC laws

- $E$ and $E_k$ are the expectation operators with respect to the along the trial and trial-to-trial domains, respectively
- $\delta x(0, k)$ and $\delta u(p, 0)$ are the initial state and input errors, respectively.

S. S. Saab
A discrete-time stochastic learning control algorithm

S. S. Saab
Stochastic P-type/D-type iterative learning control algorithms
Control Structure

Parallel arrangement of an ILC controller and a PID controller:

\[ u_k(t) = f_k(t) + [K_p e_k(t) + K_i \int_0^t e_k(\tau) d\tau + K_d \frac{de_k(t)}{dt}] \quad (12) \]

Define: \( \text{PID}[e_k(t)] \triangleq K_p e_k(t) + K_i \int_0^t e_k(\tau) d\tau + K_d \frac{de_k(t)}{dt} \)

\[ u_{k+1}(t) = f_{k+1}(t) + \text{PID}[e_{k+1}(t)] \]
\[ = f_k(t) + K_k [e_k(t + 1) - e_k(t)] + \text{PID}[e_{k+1}(t)] \]
\[ = u_k(t) + \text{PID}[e_{k+1}(t) - e_k(t)] + K_k [e_k(t + 1) - e_k(t)] \quad (13) \]
Preliminary Simulations

- Gantry robot model with a sampling time 0.01 seconds.

Tuning of the PID parameters

![Graph showing X-axis MSE (D-type Stochastic Learning) for different PID settings.

Figure 2: Representative results.
Preliminary Simulations

Tuning of $P_{u,0}$

Figure 3: Representative results. (in the results of this section ‘iteration’ is to be read as ‘trial’
Preliminary Simulations

Tuning of $Q_t$

Figure 4: Representative results.
Zero-Phase Filter Design

\[ H(z) = \frac{0.102693 + 0.002934z^{-1} + 0.002934z^{-2} + 0.102693z^{-3}}{1 - 1.644597z^{-1} + 1.091881z^{-2} - 0.236029z^{-3}} \]

Filter arrangement:

Filtering the output of ILC controller.

Filtering the error before ILC update.

Figure 5: Two possible arrangements.
Experimental Results

Experiments for the D-type stochastic algorithm.

- Varying PID parameters
  - PID = \{60, 30, 0.2\}
  - PID = \{6, 3, 0.2\}

- Varying parameter $P_{u,0}$
  - $P_{u,0} = 1000$
  - $P_{u,0} = 500$
  - $P_{u,0} = 100$

- Varying parameter $Q_t$
  - $Q_t = 10^{-1}$
  - $Q_t = 10^{-2}$
  - $Q_t = 10^{-3}$
  - $Q_t = 10^{-4}$
Experimental Results

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  - \( Q_t = 10^{-3} \)
  - \( Q_t = 10^{-4} \)
Experimental Results

Experiments for the P-type stochastic algorithm.

- **Control Input**: Input Voltage (V)
  - Time (s)
  - Displacement (m)

- **Tracking Output**: Tracking Error
  - Iteration 3
  - Iteration 5
  - Iteration 10

- **Error Frequency Spectrum**: Frequency (Hz)
  - Iteration 3
  - Iteration 5
  - Iteration 10

- **Reference** and **Output**

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Experimental Results

Experiments for the P-type stochastic algorithm.

- Varying parameter $P_{u,0}$
  - $P_{u,0} = 10$
  - $P_{u,0} = 50$
  - $P_{u,0} = 100$

- Varying parameter $Q_t$
  - $Q_t = 10^{-1}$
  - $Q_t = 10^{-2}$
  - $Q_t = 10^{-3}$

- Comparison with D-type
Experimental Results

Experiments for the P-type stochastic algorithm.

- Varying parameter $P_{u,0}$
  - $P_{u,0} = 10$
  - $P_{u,0} = 50$
  - $P_{u,0} = 100$

- Varying parameter $Q_t$
  - $Q_t = 10^{-1}$
  - $Q_t = 10^{-2}$
  - $Q_t = 10^{-3}$

- Comparison with D-type
Experimental Results

Experiments for the P-type stochastic algorithm.

- Varying parameter $P_{u,0}$
  - $P_{u,0} = 10$
  - $P_{u,0} = 50$
  - $P_{u,0} = 100$

- Varying parameter $Q_t$
  - $Q_t = 10^{-1}$
  - $Q_t = 10^{-2}$
  - $Q_t = 10^{-3}$

- Comparison with D-type

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D-type: $P_{u,0} = 500$, $Q_t = 0.001$, PID = {6,3,0.2}

P-type: $P_{u,0} = 50$, $Q_t = 0.01$, PID = {600,300,0.2}
Experimental Results

Figure 7: A comparison of Mean Square Error (MSE) against the results produced by other algorithms.
Zero-Phase Filter Design

Zero-phase low pass filter

\[ H(z) = \frac{0.102693 + 0.002934z^{-1} + 0.002934z^{-2} + 0.102693z^{-3}}{1 - 1.644597z^{-1} + 1.091881z^{-2} - 0.236029z^{-3}} \]

A comparison of the two arrangements

![Simulations (Disturbance: 0.0002)](image1)

![Experimental Results](image2)

Figure 8: Simulations and experimental results for the two filter arrangements.
An Alternative Design
ILC Problem Formulation

- **Plant Model**

\[ e_k(t) = -G(z)u_k(t) + d(t) + w_k(t) \]  \hspace{1cm} (14)

where \( d \) is a deterministic signal, \( w \) is a stationary random disturbance, and \( G \) is a stable system.

- \( d(t) \) captures trial-invariant disturbances and initial conditions
Problem Formulation

- ILC law

\[ u_{k+1}(t) = Q(z)[u_k(t) + L(z)\hat{e}_k(t)] \quad (15) \]

where \( \hat{e}_k(t) \) is the noise corrupted error measurement modeled as

\[ \hat{e}_k(t) = e_k(t) + v_k(t) \quad (16) \]

and \( v_k(t) \) is stationary random noise.
Figure 9: Block diagram representation.
Problem Formulation

- **Assumptions:**
  1. \( u_0(t) = 0 \)
  2. \(|d(t)| < M\)
  3. \( E[w_{j_1}(t_1), w_{j_2}(t_2)] = 0 \) for all \( j_1, j_2, t_1, t_2 \)
  4. \( E[w_{j_1}(t_1), w_{j_2}(t_2)] = 0, E[v_{j_1}(t_1), v_{j_2}(t_2)] = 0, E[w_{j_1}(t_1), d(t_2)] = 0, E[v_{j_1}(t_1), d(t_2)] = 0 \) for all \( j_1 \neq j_2 \) and all \( t_1, t_2 \)
  5. \( G(z) \) is a rational function with relative degree 0.

- If the relative degree is greater than zero then the analysis is easily modified.

- The spectrum of a signal, say \( r(t) \), is

\[
\Phi_r(\omega) = \sum_{\tau=-\infty}^{\infty} R_r(\tau)e^{-j\omega \tau}
\]

where the **autocorrelation function** of \( r(t) \) is

\[
R_r(\tau) = \sum_{t=0}^{\infty} E[r(t)r(t + \tau)]
\]
Power Spectrum of Asymptotic Error

- Next the power spectrum of the asymptotic error is required together with sufficient conditions for its convergence and the trial-domain convergence rate.

- Start with

\[
e_j(t) = Q(z)[1 - L(z)G(z)]e_{j-1}(z) + (1 - Q(z))d(t) + w_j(t) - Q(z)w_{j-1}(t) - Q(z)L(z)G(z)v_{j-1}(t)
\]

(17)

- It is not possible to find the power spectrum of \( e_j \) from the recursive solution of (17) since \( e_{j-1} \) and \( w_{j-1} \) are correlated.

- Instead, it can be shown that the non-recursive solution of (17) is given as follows.
Power Spectrum of Asymptotic Error

\[ e_j(t) = X_j(z)d(t) \]

\[ + \sum_{i=0}^{j-1} Y_i(z)(w_{j-1-i}(t) + v_{j-1-i}(t)) + w_j(t) \quad (18) \]

for \( j \geq 1 \) where

\[ X_j(z) = [Q(z)(1 - L(z)Q(z))]^j \]

\[ + \sum_{i=0}^{j-1} [Q(z)(1 - L(z)G(z))]^i(1 - Q(z)) \]

\[ Y_i(z) = -[Q(z)(1 - L(z)G(z))]^iQ(z)L(z)G(z) \]
Main Result

Theorem

If

\[ \max_{\omega \in [-\pi, \pi]} |Q(e^{j\omega})[1 - L(e^{j\omega})G(e^{j\omega})]| < 1 \]

the error spectrum converges and

\[ \Phi_{e_\infty}(\omega) := \lim_{j \to \infty} \Phi_{e_j}(\omega) \]

exists and is given by

\[ (W = |1 - Q(e^{j\omega})[1 - L(e^{j\omega})G(e^{j\omega})]|^2) \]

\[ \Phi_{e_\infty}(\omega) = \frac{1}{W}(|1 - Q(e^{j\omega})|^2 \Phi_d(\omega) + \Phi_w(\omega) \]

\[ + \frac{1}{W}(|Q(e^{j\omega})L(e^{j\omega})G(e^{j\omega})|^2(\Phi_w(\omega) + \Phi_v(\omega))) \]

(19)
Consider the class of model-inversion learning functions

\[ L(e^{j\omega}) = \eta(\omega)G^{-1}(e^{j\omega}) \]  \hspace{1cm} (20)

where \( \eta \) is the real valued inversion gain. Also write \( Q(e^{j\omega}) \) in Euler form

\[ Q(e^{j\omega}) = \zeta(\omega)e^{j\phi(\omega)} \]

where \( \zeta \) and \( \phi \) are real with \( \zeta \geq 0 \), and \(-\pi \leq \phi \leq \pi\). The problem now is to find the best filter design, i.e., \( \eta^*(\omega) \), \( \zeta^*(\omega) \) and \( \phi^*(\omega) \), where the superscript * denotes the complex conjugate, that minimizes the asymptotic power spectrum.

Further analysis shows that as the minimum asymptotic power spectrum is approached, the convergence rate approaches unity. Hence it is necessary to find a trade-off between asymptotic performance and convergence rate.
The optimal design problem with model-inversion learning (20) and a maximum desired convergence rate $\hat{\gamma}$ is to find $\eta^*(\omega)$, $\zeta^*(\omega)$ and $\phi^*(\omega)$ that solve

$$\min_{\eta,\zeta,\phi} \Phi_\infty(\omega) : \gamma \leq \hat{\gamma} < 1$$

(21)

where

$$\gamma = \max_{\omega \in [-\pi, \pi]} |Q(e^{j\omega})(1 - L(e^{j\omega})G(e^{j\omega}))|^2$$

(22)

The solution of this problem is known together with extensions to robustness against model uncertainty and the effects of a finite time horizon that will arise in some practical implementations of ILC.
Design

- **Implementation Issue**

- The power spectrum of $d$ is approximated using its Fourier transform and scaling of the trial length $N$ as

$$
\Phi_d^N(\omega) = \frac{1}{N} \left| \sum_{t=0}^{N-1} d(k) e^{-\frac{j\omega t}{N}} \right|^2
$$

- The optimal filter is then constructed using this last quantity.
Design

- **Practical Considerations**
- In some applications it may not be cost-effective to develop accurate noise and disturbance spectra for optimally shaping the learning and $Q$-filters. An alternative in such cases is to use simple design guidelines based on an approximate, or assumed, deterministic-stochastic ratio (DSR) defined as

  \[
  \frac{\Phi_d(\omega)}{\Phi_w(\omega) + \Phi_v(\omega)}
  \]

- **Rationale:** at frequencies where the stochastic noise is very small, there is no penalty to fast unfiltered learning and hence for large DSR set $\zeta^*(\omega) = 1$ and $\eta^*(\omega) = 1$. At frequencies where the deterministic error is very small, there is little advantage to learning and hence set $\zeta^*(\omega) = 0$ or $\eta^*(\omega) = 0$. 

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• Practical Considerations

• The advantage resulting from setting $\eta^*(\omega) \approx 0$ and keeping $\zeta^*(\omega) \approx 1$ is that the deterministic error will eventually be learned but very slowly. The advantage arising from setting $\zeta^*(\omega) \approx 0$ is improved robustness.

• These guidelines are frequency dependent and can be used to shape $\zeta(\omega)$ and $\eta(\omega)$ as the DSR changes in different frequency bands. For example, appropriate shaping of these quantities results in the well known model-inversion learning with a low-pass $Q$-filter.
Figure 10: $X$-axis component of the 3D Reference Trajectory for the gantry robot.
Experimental Application — Gantry Robot

• To obtain the disturbance model from the gantry robot, a number of zero input signals were fed into the system and the output and error signals are measured.

• When the input applied to (14) is zero

\[ e_k(t) = d(t) + w_k(t) \]

and when a number of zero input signals are applied

\[ \sum_{k=1}^{N} e_k(t) = \sum_{k=1}^{N} d(t) + \sum_{k=1}^{N} w_k(t), \]

where as \( N \to \infty \), \( \sum_{k=1}^{N} w_k(t) = 0 \).

• Hence

\[ \sum_{k=1}^{N} e_k(t) \approx N d(t), \quad d(t) \approx \frac{1}{N} \sum_{k=1}^{N} e_k(t) \quad (23) \]
Experimental Application — Gantry Robot

• On each trial

\[ w_k(t) \approx e_k(t) - d(t) \]  \hspace{1cm} (24)

and, using FFT to denote the fast Fourier transform,

\[ \Phi_w(\omega) \approx \frac{1}{N} \sum_{k=1}^{N} |\text{FFT}[w_k(t)]|^2 \]  \hspace{1cm} (25)

• Also a transfer-function, or even a constant for simplicity, can be fitted to \( \Phi_d(\omega) \) and \( \Phi_w(\omega) \). Moreover, in implementation the random noise term \( \Phi_v(\omega) \) can be assumed to be zero or a very small constant offset.
Experimental Application — Gantry Robot

Figure 11: DSR and the deterministic and stochastic spectra.
Experimental Application — Gantry Robot

• The filters that approximate the optimal $\eta^*$ and $\phi^*$ must have zero-phase and this is emulated by applying the MATLAB \textit{filtfilt} technique to a fourth-order low-pass filter to give

$$H(z) = \frac{0.0002 + 0.0007z^{-1} + 0.0011z^{-2}}{1 - 3.5328z^{-1} + 4.7819z^{-2}} \ldots + 0.0007z^{-3} + 0.0002z^{-4}$$

$$-2.9328z^{-3} + 0.6868z^{-4}$$ (26)

• To provide a comparison the trial-varying heuristic filter

$$L(z) = \frac{1}{k+1} \cdot G^{-1}(z)$$ (27)

has similar gain at low frequency but at higher frequencies the optimal filters can minimize the effect of amplifying the noise.
Experimental Application — Gantry Robot

Figure 12: MSE plots without the PID controller.
Figure 13: MSE plots with the PID controller.
Figure 14: Without (top) and with (bottom) the PID controller.
Conclusions

• As the trial number increases, the tracking error is reduced and the noise starts to dominate the signal, and the performance of optimal filters is better.

• When the PID controller is used, the noise starts to dominate from the very early stage of learning progress and the optimal filters again clearly outperforms the heuristic filter.
Figure 15: New two-input two-output system — one channel for injecting noise!!
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